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LETTER TO THE EDITOR

Renormalisation group approach to an infinite set of exponents of random resistor networks at the percolation threshold

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Abstract. A position space renormalisation group method is presented to analyse the scaling structure of the higher moments of the current distribution in random linear resistor networks at the percolation threshold. The recursion relation for the current distribution is derived under the renormalisation transformation. The current fraction assigned to each bond is represented by a random multiplicative process. An infinite set of exponents is calculated to describe the scaling properties of the current distribution.

Recently, there has been increasing interest in the critical behaviour of random resistor networks. Different properties of such systems are found to probe different critical exponents, or fractal dimensions. It has only been very recently that attention has turned to the distribution of voltage drops across each conductor in a resistor network (de Arcangelis *et al* 1985a, b). The study of percolating resistor networks led to the identification of several infinite sets of exponents, relevant to their physical properties. The infinite set of exponents provides detailed microscopic information about the structure of the network, in addition to the network conductivity. There are the two infinite sets of exponents for the higher moments of the voltage distribution on a linear resistor network and the resistance of non-linear networks (Blumenfeld *et al* 1986). In order to discuss the voltage distribution analytically, they introduced a simple hierarchical model.

In this letter, we present a renormalisation group method for the scaling structure of the current distribution on random linear resistor networks. We derive the recursion relation for the current distribution under a renormalisation transformation. The current distribution is represented by a random multiplicative process. We find the infinite set of exponents for the higher moments of the current distribution.

We restrict ourselves to the bond percolation problem on the square lattice. The lattice is divided into cells of linear dimension b, a probability p is associated with each occupied bond in the cell and the cells are rescaled to a single bond. The simplest example is indicated in figure 1(a). Each cell, divided by the broken lines, is renormalised to a single bond of linear dimension $b=2$. We note that our scale factor is different in comparison with the original decimation transformation (Young and Stinchcombe 1975, Nagatani 1986, 1987). The entries and exits of current are indicated by arrows. The probability $R(p)$ that a cell of size b is connected between the entries and the exits is given by

$$R(p)=\sum_{\alpha} f_{\alpha} \qquad (1)$$

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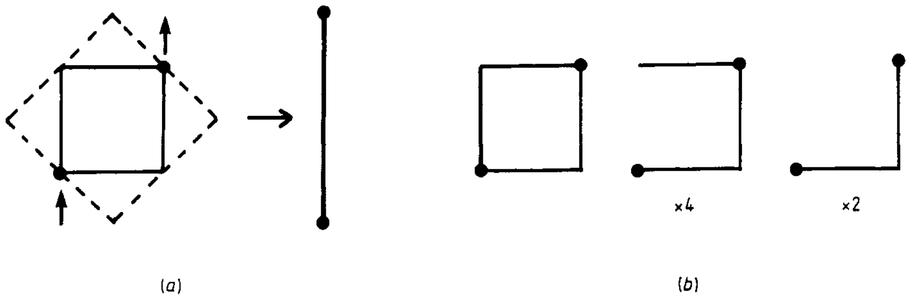

Figure 1. (*a*) Illustration of the dividing and rescaling of a $b=2$ cell for bond percolation on the square lattice. Arrows indicate the entry and exit of current. Broken lines indicate boundaries dividing the square lattice into the cells. (*b*) Spanning configurations that arise in the renormalisation group.

where the f_α is the probability of a spanning configuration α. At the fixed point $p^*=R(p^*)$, an incipient infinite cluster appears. The fractal dimensions D_b and D_c of the backbone and the cutting bonds are given by

$$D_\mathrm{b}=\ln\langle\langle n_\mathrm{b}\rangle\rangle^*/\ln b \qquad D_\mathrm{c}=\ln\langle\langle n_\mathrm{c}\rangle\rangle^*/\ln b \tag{2}$$

where the asterisk indicates the value at the fixed point and the $\langle\langle n_\mathrm{b}\rangle\rangle$ and $\langle\langle n_\mathrm{c}\rangle\rangle$ are the average numbers of backbone bonds and cutting bonds within a spanning cluster if the cell is connected. Figure 1(*b*) shows spanning configurations for a $b=2$ cell. The $\langle\langle n_\mathrm{b}\rangle\rangle$ and $\langle\langle n_\mathrm{c}\rangle\rangle$ are respectively given by

$$\langle\langle n_\mathrm{b}\rangle\rangle=(4p^4+2\times 4p^3q+2\times 2p^2q^2)/R(p)$$
$$\langle\langle n_\mathrm{c}\rangle\rangle=(2\times 4p^3q+2\times 2p^2q^2)/R(p) \tag{3}$$

where $R(p)=2p^2-p^4$.

We derive the above fractal dimensions and the infinite set of exponents from the moments of the current distribution. Each bond of the percolating network can be characterised by the fraction of the total current flowing through it, $\tilde{I}=I/I_\mathrm{tot}$. The moments of the current distribution and corresponding exponents $\tilde{\zeta}_k$ can be defined by

$$\sum_{I}\tilde{I}^k n(\tilde{I})=\left\langle\sum_{j\in\Gamma_\mathrm{b}}\tilde{I}_j^k\right\rangle\sim L^{\tilde{\zeta}_k} \tag{4}$$

where the $n(\tilde{I})$ is the number of bonds with a current fraction $\tilde{I}$, the Γ_b is the set of the backbone bonds and the $\langle\ \rangle$ represents the average.

After the renormalisation, the current fraction $\tilde{I}_j(L)$ on any backbone bond j is given by

$$\tilde{I}_j(L)=\tilde{i}_j(k)\tilde{I}_k(L/b) \tag{5}$$

where the L represents the size of the system, the b is the scale factor and the $\tilde{i}_j(k)$ indicates the current fraction of the backbone bond j within the cell k. After many repeated renormalisations at the fixed point, the relationship (5) becomes a random multiplicative process of the random variable i which is the current fraction within the cell at the fixed point. The relation (5) is not the conventional scaling relation but represents a random multiplicative process. It is the most important feature of our approach, characterising the scaling structure of the current distribution.

From (5) we can construct an infinite set of exponents $\tilde{\zeta}_k$:

$$\tilde{\zeta}_k = \log\left\langle \sum_j \tilde{I}_j^k \right\rangle (\log L)^{-1} = \log\left(\sum_\alpha C_\alpha \left(\sum_{j_\alpha} \tilde{i}_{j_\alpha}^k \right)\right)(\log b)^{-1} \tag{6}$$

where C_α is the probability of a particular spanning configuration α when the cell is connected: $C_\alpha = f_\alpha^*/R(p^*)$. For the $b=2$ case (figure 1(*b*)), we obtain

$$\tilde{\zeta}_k = \log\{[p^{*4}/R(p^*)][4(1/2)^k] + [4p^{*3}q^*/R(p^*)][2(1/1)^k] + [2p^{*2}q^{*2}/R(p^*)][2(1/1)^k]\}/\log 2. \tag{7}$$

The exponents $\tilde{\zeta}_0$ and $\tilde{\zeta}_\infty$ agree with the fractal dimensions D_b and D_c of the backbone and the cutting bonds, shown by equations (2) and (3). In figure 2, we show the plot of the exponent $\tilde{\zeta}_k$ as a function of *k*. Eliminating *L* in favour of L_1 (L_1 is the number of links), we compare $\zeta_k (= \tilde{\zeta}_k/\tilde{\zeta}_\infty)$ with the other available estimates and the hierarchical model (H model) of de Arcangelis *et al* (1985a, b) in table 1.

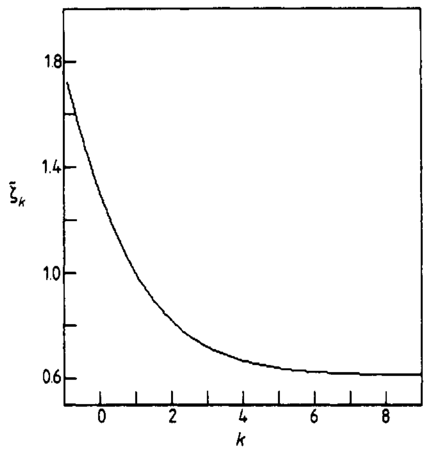

Figure 2. Plot of the exponent $\tilde{\zeta}_k$ as a function of *k*, based on the predictions of the renormalisation group method.

Table 1. List of exponents from the renormalisation group approach compared with other sources.

		ζ_k	
k	RG	Data	H model
0	2.135	2.16[a], 2.11[b]	2.00
1	1.635	1.58[a], 1.73[c]	1.585
2	1.338	1.30[a], 1.297[d]	1.322
3	1.175	1.12[a]	1.17
4	1.089	1.01[a]	1.09

[a] de Arcangelis *et al* (1985a, b).
[b] Coniglio (1982).
[c] Lyklema and Kremer (1984).
[d] Zabolitsky (1984).

To summarise, we present the renormalisation group method to derive the infinite set of exponents for the current distribution at the percolation threshold. It is shown that the current fraction assigned to each bond is represented by a random multiplicative process of the cell current fraction. The exponents $\tilde{\zeta}_0$ and $\tilde{\zeta}_\infty$ agree with the fractal dimensions of the backbone and cutting bonds, derived by the other renormalisation group method. Our renormalisation group approach is completely general and is not limited to the particular cell considered here.

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